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Size effects and micromechanics of a porous solid

RODERIC S. LAKES
Division of Materials Engineering, College of Engineering, University of Iowa, Iowa City, Iowa 52242, USA

The rigidity of rods of a polymeric foam in bending and torsion is measured as a function of diameter. The dependence of rigidity upon specimen size is found to be inconsistent with a classically viscoelastic continuum model. The Cosserat continuum, which admits additional degrees of freedom associated with rotation of the microstructure, describes the foam more accurately than the classical continuum. Evidence is presented that additional degrees of freedom, associated with the deformation of the microstructure, must be incorporated in a complete continuum model of foamed materials.

1. Introduction

For many purposes it is considered appropriate to model engineering materials as continua. The alternative of performing a detailed analysis of the transmission of forces between the individual grains of a polycrystalline material or between the constituents of a composite, is unreasonably difficult in most cases. In the classical theory of elasticity, the actual material is replaced by an equivalent continuum in which points have translational degrees of freedom only, and the transmission of load across a differential element of surface is described completely by a force vector. The predictions of elasticity theory agree with experiment for most engineering materials under most circumstances. Discrepancies have been reported in the literature between theory and experiment in fatigue properties of coarse-grained materials in regions of large strain gradient [1]. To remedy the shortcoming of classical elasticity in situations for which the microstructure size cannot be neglected in comparison with length scales of interest, several researchers have developed continuum theories for mechanical behaviour, which contain some of the degrees of freedom of the microstructure [2–8]. It is the purpose of this investigation to explore the mechanics of a foamed plastic with the aim of arriving at a generalized continuum model.

2. Experimental procedure

2.1. Quasistatic experiments

We consider a porous, polymeric foam for which the cell diameter is about one millimetre (Fig. 1). The quasistatic experimental method is based on the size effects which are predicted to occur in the bending [9] and torsion [10] of rods of a Cosserat [3] solid; the Cosserat theory is one of the simplest generalizations of classical elasticity [7]. The foam is rather compliant, therefore it is necessary to design an experimental apparatus which minimizes friction and other sources of parasitic force error. The apparatus, shown in Fig. 2, makes use of a disc-shaped permanent magnet attached to the end of a specimen to apply a torque to the specimen. The magnet is placed at the centre of a Helmholtz coil, which produces a very uniform magnetic field **B** in response to an electric current. The torque on the specimen is given by $\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$, in which $\boldsymbol{\mu}$ is the dipole moment of the magnet. The magnet is made of samarium–cobalt alloy which sustains an exceptionally high level of magnetization, or magnetic dipole moment per unit volume. The torque on the specimen is proportional to the current in the Helmholtz coil. Bending and torsion are achieved by properly orienting the Helmholtz coil with respect to the magnet, so that the torque vector is in the correct direction. The angular

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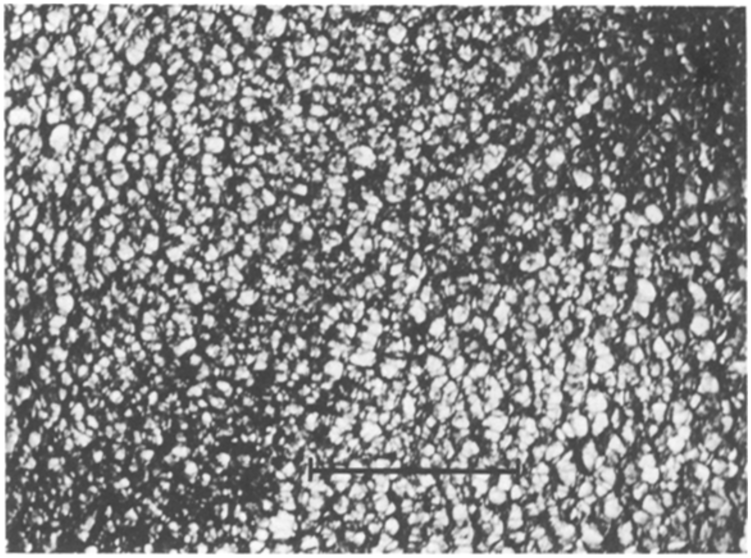

Figure 1 Optical transmission photograph of the polystyrene foam structure. Polystyrene foam; scale mark, 10 mm.

displacement of the end of the specimen is determined by measuring the linear displacement of a laser beam reflected from a mirror fixed to the specimen. The apparatus is free of hysteresis and frictional errors and can accommodate specimens with a wide range of rigidities, a necessity in size-effect studies. The apparatus rigidity is much greater than the rigidity of the largest specimen. This was verified by observing the (negligible) angular displacement of a mirror attached to the portion of the apparatus frame which supports the specimen.

Calibration of the apparatus is achieved by using an elastic (non-dissipative) specimen: a length of copper wire. The relationship between Helmholtz coil current and specimen angular displacement under static conditions is determined in torsion, then the torsional resonant frequency is determined using a sinusoidal current input to the coil. Since the metal specimen's viscoelastic loss is negligible, the torque–angle relationship is essentially independent of frequency by virtue of the Kramers–Kronig relations. The following relationship is therefore obtained:

$$\tau(i) = (2\pi\nu)^2\theta(i)I \qquad (1)$$

in which I is the mass moment of inertia of the magnet about the specimen axis, ν is the resonant frequency, τ is the torque upon the magnet due to the field of the Helmholtz coil, θ is the angle as a function of current obtained from the quasi-static test, and i is the coil current. The magnet mass, needed to compute I, is determined by means of a non-magnetic balance. Care was also taken during the weighing procedure to neutralize most the magnet's external field by clamping it with permeable steel; the mass of the clamp was then subtracted. Apparatus calibration was verified by using it to examine a rod of polymethyl methacrylate, which has known properties and which obeys classical viscoelasticity on a macroscopic scale.

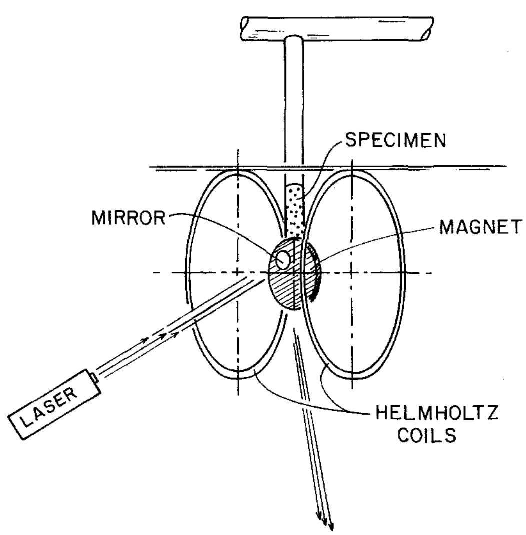

Figure 2 Experimental apparatus.

A foam specimen was prepared by rough-cutting a rectangular block into an octagonal prism and shaping it into circular cylindrical form on a lathe. The specimen was then cemented to polymethyl methacrylate end plates with a cyanoacrylate adhesive, and mounted in the apparatus. It was tested in bending and torsion as described above; step current, therefore step torque, was applied. The foam is viscoelastic, therefore isochronal data were extracted, at a time 15 seconds after the application of the step torque. Simple tension experiments were also performed in creep mode by means of dead weights. Specimen end displacement was measured without contact, in this case using a microscope with a calibrated reticle. Poisson contraction was observed using the same microscope to observe motion of wire markers glued to the lateral surface of the specimen. Again, isochronal data at time 15 sec after loading, were obtained. Following this procedure, the specimen mass and dimensions were determined, and it was cut down to a smaller size so that the length/diameter ratio was held constant.

2.2. Dynamic experiments

The dispersive behaviour of the foam was examined in free–free unloaded resonance vibration experiments upon rectangular bars of foam. Longitudinal vibrations were excited in the foam by an electromagnetic technique in which an oscillatory current passed through several parallel fine (no. 38), i.e. 0.10 mm diameter, wires cemented to one end of the specimen and immersed in a static magnetic field. The vibrations were detected using a similar apparatus at the other end of the specimen. The excitation wave form was a series of gated bursts of sinusoids; measurements were made of the detected free-decay waveforms. To ascertain the possible effect of mass-loading by the wire assembly, the following test was performed. The bar resonant frequency was measured using assemblies of seven wires and one wire; the difference in resonant frequency was less than 1%. The experimental configuration, therefore, provides a good approximation to the ideal case of free–free resonance. The method was used to obtain the resonant frequency as a function of bar length. In the lowest mode of vibration, the bar length is one half wavelength, so a dispersion curve, i.e. the relation between frequency and wavelength, can be obtained.

3. Generalized continua

3.1. Continuum models of structured materials

If the structure of a material is sufficiently regular, or if it has appropriate statistical characteristics, it may be possible to transform the structure into an equivalent continuum. The nature of this continuum depends not only on the original structure, but also on the accuracy of the transformation, i.e. the extent to which the degrees of freedom of the structure are incorporated into the continuum model. The simplest type of continuum model of a structured material is composite theory, in which the composite material is transformed into a classically elastic, possibly anisotropic, continuum. If the physical dimensions of interest exceed the microstructure size by several orders of magnitude, classical elasticity is considered to be adequate. If not, a more general continuum model may be needed. Several such models which have attracted considerable attention, are considered in the following sections.

3.2. The Cosserat continuum

The Cosserat brothers [3] in 1909 advanced a theory for deformable bodies which admits degrees of freedom not present in classical elasticity. For example, points are considered to rotate with respect to each other in addition to undergoing translation. In recent years several researchers have explored generalized continuum theories of the Cosserat type [5–8]. The constitutive equations of linear isotropic micropolar elasticity, considered to be identical to the Cosserat elasticity are [7]:

$$\sigma_{kl} = \lambda e_{rr}\delta_{kl} + (2\mu + \kappa)e_{kl} + \kappa\epsilon_{klm}(r_m - \phi_m) \tag{2}$$

$$m_{kl} = \alpha\phi_{r,r}\delta_{kl} + \beta\phi_{k,l} + \gamma\phi_{l,k}, \tag{3}$$

in which the usual Einstein summation convention is used and the comma denotes differentiation with respect to spatial variables. e is the usual small strain tensor, defined in terms of the displacement, u: $e_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k})$; r_m is the macrorotation $r_m = \frac{1}{2}\epsilon_{mln}u_{n,l}$ in which ϵ is the permutation symbol, σ is the asymmetric force stress tensor, m is the couple stress tensor and ϕ is the microrotation vector, which is kinematically independent of the macrorotation, r. The quantities λ, μ, κ, α, β, γ are Cosserat elastic constants. Thus the isotropic Cosserat solid has six elastic constants, in contrast to the classically elastic solid which has two. The classical solid is obtained as a special case of the Cosserat solid if α, β, γ, and κ are zero. In this article we use the above notation which is due to Eringen [7]; a comparison with symbols used by other authors is given in [8].

A Cosserat elastic solid is predicted to behave differently in many respects from a classically elastic solid. In a Cosserat solid, dispersion of transverse waves [7], the existence of new kinds of waves [7], size effects in the torsion [10] and bending [9] of rods, modification in the concentration of stress around holes [11, 12] and a modification of the mode structure of vibrating bodies [13] are predicted to occur.

Despite the differences between the predictions of Cosserat and classical elasticity, few experiments have been done to determine whether a real material is describable as a Cosserat continuum. Metals, examined for possible couple stress elasticity, a special case of Cosserat theory, were found to behave classically [13, 14]. A model composite was tested to determine its micropolar elastic constants, but it was found to behave classically [10]. The same composite, studied by wave propagation methods, exhibited some nonclassical, and possibly micropolar, effects [11]. A foam material was found to exhibit size effects consistent with couple stress theory [16]. Optical studies on a molecular crystal, interpreted in light of generalized continuum theory, revealed nonclassical effects [17]. In the above experiments, no Cosserat elastic effects were found [10, 14] or rather small and subtle effects were found [15, 17] and/or viscoelastic effects were not decoupled from the sought-after geometrical effects [16]. Recently the present author has obtained evidence of Cosserat elastic behaviour in human compact bone, a natural composite [18–23]. As anticipated on the basis of continuum models of structured solids, the Cosserat characteristic lengths are comparable to the size of the dominant structural elements.

3.3. Microstructure elasticity and its special cases

The theory of elasticity with microstructure, [24] also known as micromorphic elasticity, [4] incorporates many of the degrees of freedom of a crystal lattice and is considered to be applicable to a variety of periodic and quasi-periodic structures. In this theory, each point in the continuum is endowed with a triad of director vectors, which can rotate with respect to nearby triads as in Cosserat theory, and which can deform as well. Thus, sufficient degrees of freedom are incorporated in the theory to account for acoustic wave dispersion and optical modes of lattices and other periodic structures. Eighteen independent elastic constants are required to characterize an isotropic material with microstructure, in contrast to six for a Cosserat solid and two for a classically elastic material. A solid with microstructure may be distinguished from a Cosserat solid by the dispersion of longitudinal waves predicted to occur in the former but not in the latter.

Special types of elastic material with microstructure include the Cosserat solid and the elastic solid with voids [25, 26]. In the former case, the director triad is rigid and has only the rotational degrees of freedom, while in the latter, the triad undergoes dilatation associated with pore volume changes, but there are no rotational degrees of freedom.

4. Experimental results

4.1. Size effects

To display the size-effect data, it is convenient to plot rigidity divided by the square of the diameter against the square of the diameter. Figs. 3 and 4 show the size-effect in rods of foam in torsion and bending respectively, based on isochronal data taken 15 sec after application of a step torque. For a classically elastic material, the plot should be a straight line through the origin, since the rigidity depends on the fourth power of the diameter in both torsion and bending. For a classically viscoelastic material, the plot will also be a straight line through the origin if isochronal data are used. The observed size effects are not consistent with classical elasticity or viscoelasticity. They are not a result of an artifactual densification of the spectrum since the specimen density is $36.74 \pm 0.64\,\text{kg m}^{-3}$ (mean $\pm$ standard deviation) and the density does not exhibit any significant dependence on specimen diameter.

Analysis of these results on the basis of Cosserat theory as a generalized continuum model is as follows. Within the framework of Cosserat theory, one can express the results both in terms of the micropolar elastic constants λ, μ, κ, α, β, γ in the constitutive equations, Equations 2 and 3, and the micropolar technical constants E, G, ν, Ψ, l_t, l_b, and N. The latter are defined as follows:

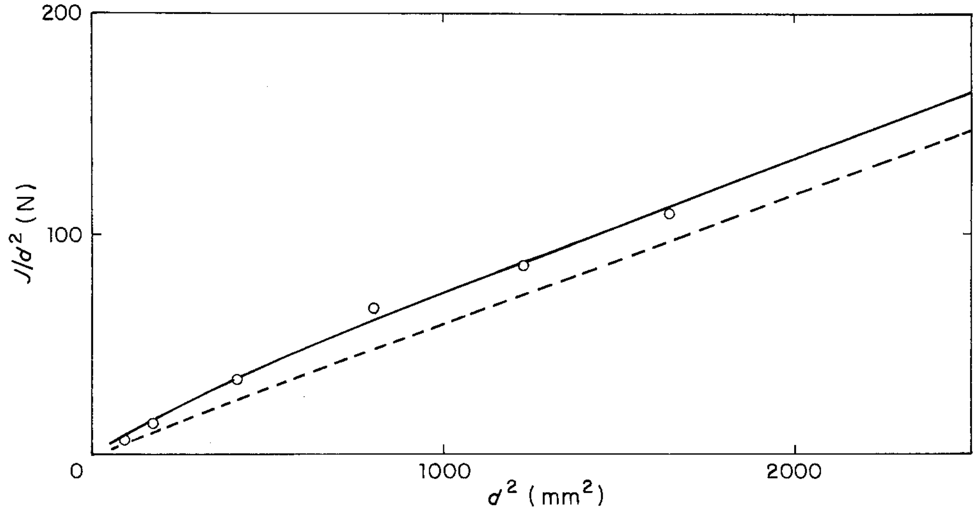

Figure 3 Experimental results, torsion. Rigidity/diameter squared against diameter squared in mm². Points are experimental; curve is micropolar, $G = 0.6\ \text{MN m}^{-2}$, $\Psi = 1.5$, $N^2 = 0.09$, $l_t = 3.8$ mm.

Young's modulus (N m^{-2})	$E = (2\mu + \kappa)(3\lambda + 2\mu + \kappa)/(2\lambda + 2\mu + \kappa)$
Shear modulus (N m^{-2})	$G = (2\mu + \kappa)/2$
Poisson ratio (dimensionless)	$\nu = \lambda/(2\lambda + 2\mu + \kappa)$
Characteristic length, torsion (m)	$l_t = [(\beta + \gamma)/(2\mu + \kappa)]^{1/2}$
Characteristic length, bending (m)	$l_b = [\gamma/2(2\mu + \kappa)]^{1/2}$
Coupling number (dimensionless)	$N = [\kappa/2(\mu + \kappa)]^{1/2}$
Polar ratio (dimensionless)	$\Psi = (\beta + \gamma)/(\alpha + \beta + \gamma)$

The relationship between various features of size effect plots for torsion [10] and bending [9] and the micropolar constants is shown in Figs. 5 and 6 respectively. In the case of torsion, (Fig. 3), the experimental data are fitted well by $G = 0.6\ \text{MN m}^{-2}$, $\Psi = 1.5$, $l_t = 3.8$ mm, and $N^2 = 0.09$. In tension, the measured Young's modulus is $E = 1.3\ \text{MN m}^{-2}$, and $\nu = 0.07$. Since $E = 2G(1 + \nu)$ as in the classical case, E based on G and ν should

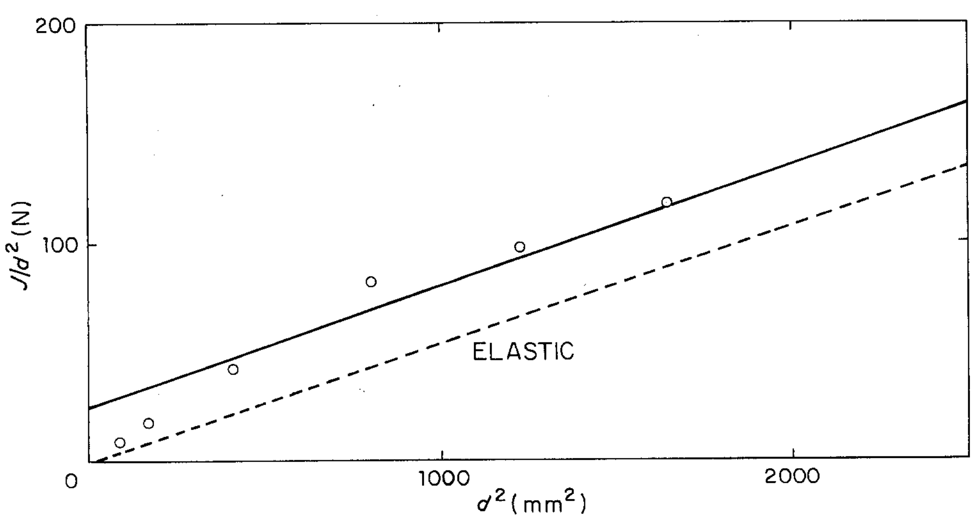

Figure 4 Experimental results, bending. Rigidity/diameter squared against diameter squared in mm². Points are experimental, curve is micropolar, $E = 1.1\ \text{MN m}^{-2}$, $N^2 = 0.09$, $\beta/\gamma = -0.62$, $l_b = 5.0$ mm.

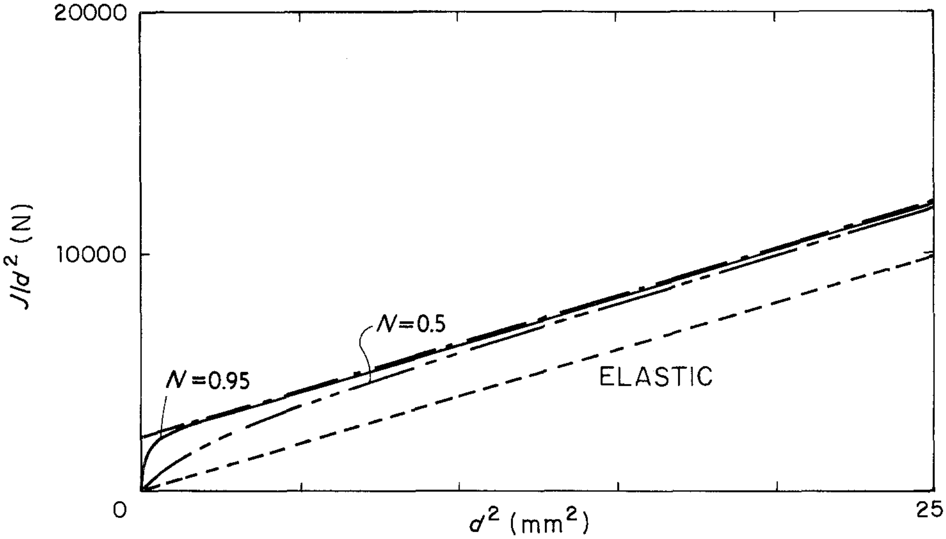

Figure 5 Torsion of a micropolar rod, predicted behaviour. J is the specimen rigidity and d is its diameter. The asymptotic slope for large d is proportional to G. Arbitrary units of force on the ordinate, arbitrary units of length squared on the abscissa.

be 1.284 MN m^{-2}, a satisfactory agreement with E measured directly in tension. In the case of bending, (Fig. 4), the data cannot be fitted accurately by a theoretical micropolar curve. For initial guesses based on the torsion results, the data points for the three largest specimens can be fitted reasonably well by $E = 1.1$ MN m^{-2}, $N^2 = 0.09$, $l_b = 5.0$ mm, and $\beta/\gamma = 0.62$, but not those with the smallest diameter. These deviations from the predictions of Cosserat theory suggest that the solid has additional degrees of freedom. We therefore examine other generalized continuum theories for their potential applicability.

The continuum theory of materials with voids [25] is of the same order of complexity as Cosserat theory, but it deals with changes in void volume as an additional kinematical variable, rather than the local rotations considered in Cosserat theory. This theory predicts size effects to occur in bending [26], but not under a state of pure shear stress associated with torsion. We observe significant size effects in torsion (Fig. 2), therefore we shall not consider the void theory further for this foam material. It may, however, be applicable to materials with a different structure.

4.2. Wave dispersion

The theory of elastic materials with microstructure is more general than either Cosserat theory or the theory of materials with voids. It is sufficiently complex that very few analyses have been carried out using it [27]. The problems of torsion and

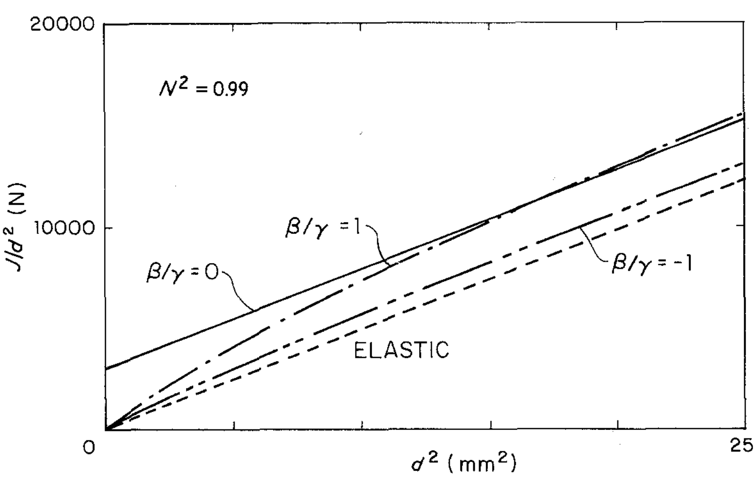

Figure 6 Bending of a micropolar rod, predicted behaviour. J is the specimen rigidity and d is the diameter. The asymptotic slope for large d is proportional to E. Arbitrary units of force on the ordinate, arbitrary units of length squared on the abscissa.

bending have not yet been solved in this theory, so that it cannot be used in the analysis of the present quasistatic experimental results. Several wave propagation problems have, however, been examined in both Cosserat and microstructure theory. In an elastic material with microstructure, both transverse and longitudinal waves are dispersive [24] while in a Cosserat solid only the transverse waves are dispersive [7]. Experiments involving longitudinal standing waves have therefore been performed. The results are shown in Fig. 7. The longitudinal waves are indeed dispersive and the form of the dispersion is in agreement with the predictions of microstructure elasticity. The physical origin of this dispersion is considered to be coupling of the acoustic wave motion to microvibrations of the unit cell of the structure. A dimensionless technical constant may be defined as follows:

$$\Lambda^2 \equiv \omega^2 \rho d^2/3E \qquad (4)$$

in which ω is the cut-off angular frequency, ρ is density, d is the cell size and E is Young's modulus.

For $\omega/2\pi = 6$ kHz, $d = 1$ mm, $\rho = 36.74$ kg m^{-3}, and $E = 1.1$ MN m^{-2}, we obtain $\Lambda = 0.13$. Since neither a classical nor a Cosserat material is predicted to exhibit any cut-off frequency for longitudinal acoustic waves, these special cases of a material with microstructure correspond to an infinite value of Λ.

5. Discussion

The results presented in the previous section indicate that the foam does not obey the classical theory of elasticity or viscoelasticity. Cosserat theory describes the size-effect behaviour in torsion and over a range of specimen sizes in bending; and for sufficiently long longitudinal waves. The wave dispersion behaviour for the full range of wavelengths is consistent with the theory of elastic materials with microstructure, a generalization of Cosserat theory. Microstructure elasticity, however, has not yet been applied to the torsion and bending geometries used in the present size-effect studies.

Very few experimental studies seeking to explore structure-related generalized continuum mechanics of materials have been performed. In most such studies the material, whether it be a polycrystalline metal [14] or a special composite [10], is found to behave classically. Recently the present author has found evidence for generalized continuum effects in a natural composite, human compact bone. In quasistatic torsion [21] and in torsion resonance [20] the observed size-effects are consistent with a restricted form of Cosserat viscoelasticity known as couple-stress theory, for specimens greater than 1 mm in diameter. The special case corresponds to $N = 1$, its maximum value. In the torsion of microsamples smaller than 1 mm [23] and in quasistatic bending, [22], the observed size effects are consistent with Cosserat theory. The experiments permit the determination of N as well as the characteristic lengths. In quasistatic experiments, human compact bone appears to obey Cosserat theory. There remains the possibility that additional degrees of freedom,

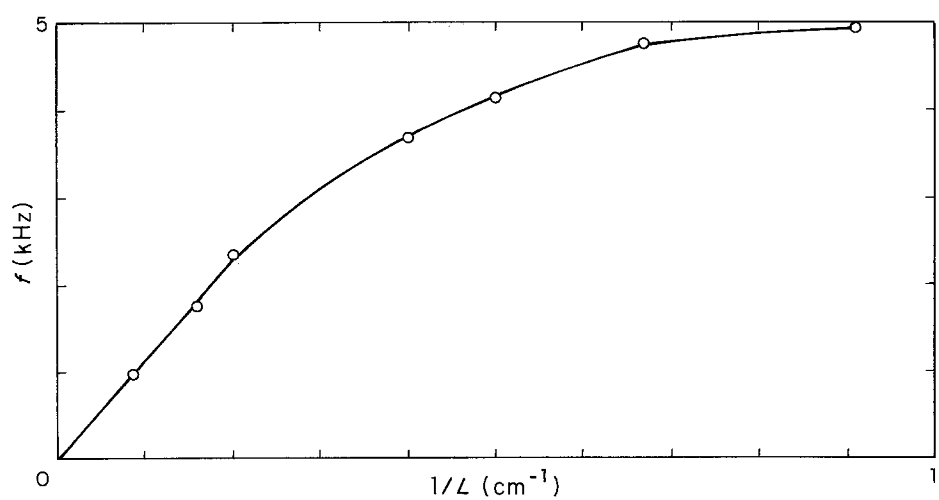

Figure 7 Dispersion of standing waves in a bar of foam: fundamental resonant frequency f against inverse of the wave length. In the free–free resonant configuration used here, the bar length is one half the wavelength.

perhaps related to microstructure elasticity, may be revealed in wave-propagation studies. We therefore examined published results on bone to get an estimate of its possible micromorphic (microstructure) elastic character. Ultrasonic longitudinal waves of up to 10 MHz have been successfully passed through bone, with no evidence of internal resonance [28] or of dispersion of the sort shown in Fig. 7. Since the density is 2×10^3 kg m^{-3} and the structure size (diameter of osteons) is 0.25 mm, we obtain $\Lambda^2 > 4.6$. The Cosserat triad appears to be rather rigid. This crude estimate of its rigidity is consistent with our previous successful application of Cosserat elasticity (with rigid directors) to bone.

A structural perspective could be taken in the analysis of size-effects and wave motion in structured materials such as foamed plastics. For example, a finite-element analysis might be performed, considering the plastic material to be classically viscoelastic and the pore space to have zero stiffness. The largest specimens examined in this study are 40 mm in diameter and 200 mm long and therefore contain about 250 000 cells. In a finite element analysis, each cell would be assigned at least four elements, or 10^6 elements in total. This greatly exceeds the 10^4 elements currently associated with "large" problems in finite element analysis. The generalized continuum approach used in this article, by contrast, is considerably simpler.

6. Conclusions

1. The porous polymeric foam of cell size ~ 1 mm examined in this study deviates from classical elasticity. A deviation of 10% or less from classical theory in torsion is expected only for diameters greater than 60 mm.

2. Cosserat elasticity is an accurate continuum model for both torsion and bending of specimens 28 to 40 mm in diameter, and presumably for larger specimens. The torsion behaviour is well-described by Cosserat theory for the full range of specimen sizes. Wave experiments are consistent with Cosserat theory only for wavelengths greater than 100 mm.

3. A more general continuum approach is necessary to model the full range of observed behaviour. The theory of elasticity with microstructure, or the micromorphic theory, should be an appropriate generalization of Cosserat elasticity suitable for the solid examined here. Micromorphic degrees of freedom may be responsible for the quasistatic results in the bending of thin rods.

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